

ABSOLUTE CONTINUITY, INTERPOLATION AND THE LYAPUNOV ORDER

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ABSTRACT. We extend our Nevanlinna-Pick theorem for Hardy algebras and their representations to cover interpolation at the absolutely continuous points of the boundaries of their discs of representations. The Lyapunov order plays a crucial role in our analysis.

1. INTRODUCTION

The celebrated theorem of Nevanlinna and Pick asserts that if n distinct points, z_1, z_2, \dots, z_n , are given in the open unit disc \mathbb{D} and if n other complex numbers are also given, w_1, w_2, \dots, w_n , then there is a function f in the Hardy algebra $H^\infty(\mathbb{T})$, with norm at most one, such that $f(z_i) = w_i$, $i = 1, 2, \dots, n$, if and only if the Pick matrix

$$\left(\frac{1 - w_i \overline{w_j}}{1 - z_i \overline{z_j}} \right)_{i,j=1}^n$$

is positive semidefinite. In [4, Theorem 5.3], we generalized this Nevanlinna-Pick theorem to the setting of Hardy algebras over W^* -correspondences. Here we intend to push the work in [4] further, using tools developed in [2]. In a sense that we shall make precise, we show that there is a condition similar to the positivity of the Pick matrix that allows one to interpolate at “absolutely continuous” points of the boundaries of the domains considered in [4]. Before stating our main theorem, we want to see how one might try to extend the Nevanlinna-Pick theorem to cover families of absolutely continuous contraction operators along the lines suggested by [4, Theorem 6.1].

For this purpose, suppose H is a Hilbert space and $Z := (Z_1, Z_2, \dots, Z_n)$ is an n -tuple of operators in $B(H)$. Then Z defines a completely positive operator Φ_Z on the $n \times n$ matrices over $B(H)$ via the formula

$$\begin{aligned} \Phi_Z((a_{ij})) &:= \begin{bmatrix} Z_1 & & & \\ & Z_2 & & \\ & & \ddots & \\ & & & Z_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} Z_1 & & & \\ & Z_2 & & \\ & & \ddots & \\ & & & Z_n \end{bmatrix}^* \\ &= (Z_i a_{ij} Z_j^*). \end{aligned}$$

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If all the Z_i 's have norm less than 1, then $I - \Phi_Z$ is an invertible map on $M_n(B(H))$ and $(I - \Phi_Z)^{-1}$ is also completely positive. The following theorem, then, is a special case of [4, Theorem 6.1].

Theorem 1.1. *Suppose Z_1, Z_2, \dots, Z_n are n distinct operators in $B(H)$, each of norm less than 1, and suppose W_1, W_2, \dots, W_n are n operators in $B(H)$. Then there is a function f in $H^\infty(\mathbb{T})$, with supremum norm at most 1, such that $f(Z_i) = W_i$, $i = 1, 2, \dots, n$, where $f(Z_i)$ is defined through the Riesz functional calculus, if and only if the Pick map*

$$(1) \quad (I - \Phi_W) \circ (I - \Phi_Z)^{-1}$$

defined on $M_n(B(H))$ is completely positive.

Observe that when H is one-dimensional this theorem recovers the classical theorem of Nevanlinna and Pick that we stated at the outset. Now Sz.-Nagy and Foias have shown that the proper domain for their H^∞ -functional calculus is the collection of all *absolutely continuous contractions*. One way to say that a contraction T is absolutely continuous is to say that when T is decomposed as $T = T_{cnu} + U$, where T_{cnu} is completely non-unitary and U is unitary, then the spectral measure of U is absolutely continuous with respect to Lebesgue measure on the circle. The content of [10, Theorems III.2.1 and III.2.3] is that a contraction T is absolutely continuous if and only if the $H^\infty(\mathbb{T})$ -functional calculus may be evaluated on T . It is therefore of interest to modify the hypothesis of Theorem 1.1 and ask for conditions that allow one to interpolate in the wider context where the variables Z_i are assumed to be merely absolutely continuous contractions. In that setting the map $I - \Phi_Z$ need no longer be invertible, and so it may not be possible to form the generalized Pick operator $(I - \Phi_W) \circ (I - \Phi_Z)^{-1}$, let alone determine whether or not it is completely positive. However, there is a notion from matrix analysis, called the ‘‘Lyapunov order’’, which suggests a replacement for the condition that the Pick operator $(I - \Phi_W) \circ (I - \Phi_Z)^{-1}$ be completely positive. To formulate it we require an idea from the theory of completely positive maps that we analyzed in [2].

Definition 1.2. Let Φ be a completely positive map on a W^* -algebra A . An element $a \in A$ is called *superharmonic* for Φ in case $a \geq 0$ and $\Phi(a) \leq a$; a is called *pure superharmonic* in case a is superharmonic and $\Phi^n(a) \searrow 0$ as $n \rightarrow \infty$.

The superharmonic elements for a completely positive map evidently form a convex subset in the cone of all non-negative elements in the W^* -algebra A .

Definition 1.3. Let B be a W^* -algebra and suppose A is a sub- W^* -algebra of B . Suppose $\Phi : A \rightarrow A$ is a completely positive map and that $\Psi : B \rightarrow B$ is also completely positive. Then we say Ψ *completely dominates* Φ in the sense of Lyapunov in case every pure superharmonic element of $M_n(A)$ for Φ_n is superharmonic for Ψ_n , where Φ_n (resp. Ψ_n) is the usual promotion of Φ (resp. Ψ) to $M_n(A)$ (resp. $M_n(B)$).

The following proposition links the notion of complete Lyapunov domination to the complete positivity of (1).

Proposition 1.4. *Suppose that A is a sub- W^* -algebra of a W^* -algebra B and suppose Φ and Ψ are completely positive maps on A and B , respectively. Assume that $\|\Phi\| < 1$, so $I - \Phi$ is invertible. Then the Pick operator, $P := (I - \Psi) \circ (I -$*

$\Phi)^{-1}$, is completely positive if and only if Ψ completely dominates Φ in the sense of Lyapunov.

Proof. Note that the hypothesis that $\|\Phi\| < 1$ implies that every superharmonic element of A is pure superharmonic. Also note that it suffices to prove that P is positive if and only if $\{a \in A \mid a \geq 0, \Phi(a) \leq a\} \subseteq \{b \in B \mid b \geq 0, \Psi(b) \leq b\}$, since the same argument will work for every n . Suppose, then, that P is positive and suppose that $a \geq 0$ and $\Phi(a) \leq a$. Then $(I - \Phi)(a) \geq 0$. Consequently, $0 \leq P((I - \Phi)(a)) = (I - \Psi)(a)$, showing that $a \geq \Psi(a)$. Suppose, conversely, that $b \geq 0$. Then since $\|\Phi\| < 1$ and Φ is positive, $(I - \Phi)^{-1} = \sum_{n \geq 0} \Phi^n$ is positive. Consequently, $a = (I - \Phi)^{-1}(b)$ is positive. But also, since $(I - \Phi)(a) = b$ is positive, $a \geq \Psi(a)$, by hypothesis. That is, $P(b) = a - \Psi(a) \geq 0$, which is what we want to show. \square

Our extension of Theorem 1.1 can now be formulated as

Theorem 1.5. *Suppose Z_1, Z_2, \dots, Z_n are n distinct absolutely continuous contractions on a Hilbert space H and suppose W_1, W_2, \dots, W_n are n contractions on H , then there is a function $f \in H^\infty(\mathbb{T})$, of norm at most 1, such that $f(Z_i) = W_i$, $i = 1, 2, \dots, n$, if and only if Φ_W completely dominates Φ_Z in the sense of Lyapunov.*

The technology we use to prove Theorem 1.5 works in the more general context of Hardy algebras over W^* -correspondences, as we mentioned earlier. This is the arena in which our analysis takes place. But first, we must provide some background from [4, 2]. We shall follow terminology and most of the notation from [2]. In particular, we shall cite the second section of [2] for further background because it gives a fairly detailed birds-eye view of the theory as of 2010.

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2. BACKGROUND AND THE MAIN THEOREM

Throughout this note, M will denote a fixed W^* -algebra. We will treat M as an abstract C^* -algebra that is a dual space and we will not think of it as acting concretely on Hilbert space except through representations that we will specify. Also, E will denote a W^* -correspondence over M . This means first that E is a right Hilbert C^* -module over M which is self-dual. Consequently, the algebra of all bounded adjointable M -module maps on E , $\mathcal{L}(E)$, is all the bounded module maps and $\mathcal{L}(E)$ is a W^* -algebra. To say that E is a W^* -correspondence over M means, then, that there is a normal representation $\varphi : M \rightarrow \mathcal{L}(E)$, making E a left M -module [2, Paragraph 2.2]. To eliminate technical digressions we assume that φ is faithful and unital. The tensor powers of E , $E^{\otimes n}$, will be the self-dual completions of the usual C^* -Hilbert module tensor powers, and the Fock space $\mathcal{F}(E)$ will be the self-dual completion of the C^* -direct sum of the $E^{\otimes n}$. Then $\mathcal{F}(E)$ is a W^* -correspondence over M and we denote by φ_∞ the left action of M on $\mathcal{F}(E)$ [2, Paragraph 2.7]. If $\xi \in E$, then T_ξ will denote the creation operator it determines: $T_\xi \eta := \xi \otimes \eta$, $\eta \in \mathcal{F}(E)$. The norm-closed subalgebra generated $\varphi_\infty(M)$ and $\{T_\xi \mid \xi \in E\}$ is called the *tensor algebra* of E and will be denoted by

$\mathcal{T}_+(E)$ [2, Paragraph 2.7]. The ultra weak closure of $\mathcal{T}_+(E)$ in $\mathcal{L}(\mathcal{F}(E))$ is called the *Hardy algebra of E* and is denoted $H^\infty(E)$ [2, Definition 2.1].

Suppose $\sigma : M \rightarrow B(H_\sigma)$ is a normal representation and let $\sigma^E : \mathcal{L}(E) \rightarrow B(E \otimes_\sigma H_\sigma)$ be the induced representation of $\mathcal{L}(E)$ in the sense of Rieffel [7, 8]: $\sigma^E(T) := T \otimes I$, $T \in \mathcal{L}(E)$. We write $\mathcal{I}(\sigma^E \circ \varphi, \sigma)$ for the set of all operators $C : E \otimes_\sigma H_\sigma \rightarrow H_\sigma$ that satisfy the equation $C\sigma \circ \varphi(a) = \sigma(a)C$ for all $a \in M$; i.e., $\mathcal{I}(\sigma^E \circ \varphi, \sigma)$ denotes all the *intertwiners* of $\sigma^E \circ \varphi$ and σ . Also, we write $\mathbb{D}(E, \sigma)$ for the set of all elements of $\mathcal{I}(\sigma^E \circ \varphi, \sigma)$ that have norm less than 1, and we write $\overline{\mathbb{D}(E, \sigma)}$ for its norm closure. In [6] we proved

Lemma 2.1. (See [2, Paragraph 2.8].) *Given $\mathfrak{z} \in \overline{\mathbb{D}(E, \sigma)}$, define $\mathfrak{z} \times \sigma$ by $\mathfrak{z} \times \sigma(\varphi_\infty(a)) := \sigma(a)$ and $\mathfrak{z} \times \sigma(T_\xi)(h) := \mathfrak{z}(\xi \otimes h)$, $a \in M$, $\xi \in E$, and $h \in H_\sigma$. Then $\mathfrak{z} \times \sigma$ extends to a completely contractive (c.c.) representation of $\mathcal{T}_+(E)$ on H_σ . Conversely, given a c.c. representation ρ of $\mathcal{T}_+(E)$, then $\rho = \mathfrak{z} \times \sigma$, where $\sigma := \rho \circ \varphi_\infty$ and $\mathfrak{z}(\xi \otimes h) := \rho(T_\xi)h$. Further, for $F \in H^\infty(E)$, the $B(H_\sigma)$ -valued function \widehat{F}_σ , defined on $\mathbb{D}(E, \sigma)$ by $\widehat{F}_\sigma(\mathfrak{z}) := \mathfrak{z} \times \sigma(F)$, is bounded analytic and it extends to be continuous on $\overline{\mathbb{D}(E, \sigma)}$ when $F \in \mathcal{T}_+(E)$.*

Remark 2.2. We note here that our $\mathbb{D}(E, \sigma)$ is denoted $\mathbb{D}(E^\sigma)^*$ in [2, Paragraph 2.8], where $E^\sigma := \mathcal{I}(\sigma^E \circ \varphi, \sigma)^* = \mathcal{I}(\sigma, \sigma^E \circ \varphi)$ is the σ -dual of E [2, Paragraph 2.6]. This dual space plays an important role in our theory, as we shall see, but we have opted for the change of notation in order to eliminate numerous unnecessary and often confusing adjoints from our formulas.

Definition 2.3. A point $\mathfrak{z} \in \overline{\mathbb{D}(E, \sigma)}$ and the representation $\mathfrak{z} \times \sigma$ are called *absolutely continuous* in case $\mathfrak{z} \times \sigma$ extends to be an ultra weakly continuous representation of $H^\infty(E)$ in $B(H_\sigma)$. We write $\mathcal{AC}(E, \sigma)$ for all the absolutely continuous points of $\overline{\mathbb{D}(E, \sigma)}$.

Our choice of terminology is inspired by the fact that when $M = E = \mathbb{C}$, then \mathfrak{z} is absolutely continuous in our sense if and only if \mathfrak{z} , which is just an ordinary contraction operator on H_σ , is absolutely continuous in the sense described in the Introduction.

In general, $\mathbb{D}(E, \sigma) \subseteq \mathcal{AC}(E, \sigma) \subseteq \overline{\mathbb{D}(E, \sigma)}$, and both inclusions are proper. If $M = E = \mathbb{C}$, and if σ is the one-dimensional representation of \mathbb{C} on \mathbb{C} , then $\mathbb{D}(E, \sigma)$ is the open unit disc in the complex plane and $\mathbb{D}(E, \sigma) = \mathcal{AC}(E, \sigma)$. In every other setting of which we are aware, $\mathbb{D}(E, \sigma) \subsetneq \mathcal{AC}(E, \sigma)$. Also, we know of no situation where $\overline{\mathbb{D}(E, \sigma)} = \mathcal{AC}(E, \sigma)$. We have been able to identify $\mathcal{AC}(E, \sigma)$ explicitly in numerous instances [2, Sections 4 and 5] and we know a lot about this space, but there is still much that remains mysterious.

The σ -dual of E , $E^\sigma := \mathcal{I}(\sigma, \sigma^E \circ \varphi)$, is important in this study for several reasons. The first is that it is a W^* -correspondence over $\sigma(M)'$ in a very natural way. For $\xi, \eta \in E^\sigma$, $\langle \xi, \eta \rangle$ is defined to be $\xi^* \eta$ - the product being the ordinary operator product, which makes sense as an operator on H_σ since ξ and η both map from H_σ to $E \otimes_\sigma H_\sigma$. The actions of $\sigma(M)'$ on E^σ are given by the formula:

$$a \cdot \xi \cdot b := (I_E \otimes a)\xi b, \quad a, b \in \sigma(M)', \xi \in E^\sigma.$$

Again, the products on the right hand side of the equation are ordinary operator products. The concept of the σ -dual of a W^* -correspondence was formalized in [4], but it appeared, implicitly, in a number of places. A key role that it will play

here is in the identification of the commutant of an induced representation, which we will describe in the next section. But here we can already see its relevance for the present considerations by virtue of the following observation: Let $\mathfrak{z}_1, \mathfrak{z}_2, \dots, \mathfrak{z}_n$ be points in $\overline{\mathbb{D}(E, \sigma)}$. Then they define a map $\Phi_{\mathfrak{z}}$ on the $n \times n$ matrices over $\sigma(M)'$ by the formula

$$(2) \quad \Phi_{\mathfrak{z}}((a_{ij})) := (\langle \mathfrak{z}_i, a_{ij} \cdot \mathfrak{z}_j \rangle) \quad (a_{ij}) \in M_n(\sigma(M)').$$

A moment's reflection reveals that $\Phi_{\mathfrak{z}}$ is completely positive, as it is the composition of manifestly completely positive maps.

Our objective in this note is the proof of the following theorem, which will occupy the next section.

Theorem 2.4. *Suppose E is a W^* -correspondence over a W^* -algebra M and that σ is a faithful normal representation of M on the Hilbert space H_σ . Suppose, too, that n distinct points $\mathfrak{z}_1, \mathfrak{z}_2, \dots, \mathfrak{z}_n \in \mathcal{AC}(E, \sigma)$ are given and that n operators in $B(H_\sigma)$, W_1, W_2, \dots, W_n , are given. Define the map $\Phi_{\mathfrak{z}}$ on $M_n(\sigma(M)')$ by the formula $\Phi_{\mathfrak{z}}((a_{ij})) = (\langle \mathfrak{z}_i, a_{ij} \cdot \mathfrak{z}_j \rangle)$ and define the map Φ_W on $M_n(B(H_\sigma))$ by the formula $\Phi_W((T_{ij})) := (W_i T_{ij} W_j^*)$. Then there is an element F in $H^\infty(E)$, with $\|F\| \leq 1$, such that $\widehat{F}(\mathfrak{z}_i) = W_i$, $i = 1, 2, \dots, n$, if and only if Φ_W completely dominates $\Phi_{\mathfrak{z}}$ in the sense of Lyapunov.*

Proof of Theorem 1.5. That theorem is an immediate consequence of Theorem 2.4. Indeed, in the setting of the former, $M = E = \mathbb{C}$, and σ is just a multiple of the identity representation, the multiple being the Hilbert space dimension of H_σ . Since we may safely identify $\mathbb{C} \otimes_\sigma H_\sigma$ with H_σ , $\mathbb{D}(E, \sigma)$ may be identified with the closed unit ball in $B(H_\sigma)$, i.e., with all contractions on H_σ . As we noted, the celebrated theorems of Sz.-Nagy and Foiaş identify $\mathcal{AC}(\mathbb{C}, \sigma)$ with the absolutely continuous contractions on H_σ in the classical sense. When these identifications are made, $\Phi_{\mathfrak{z}}$ of Theorem 2.4 becomes the Φ_Z of Theorem 1.5 and, of course, the two Φ_W 's are the same. \square

3. THE PROOF OF THEOREM 2.4

We will first show that if Φ_W completely dominates $\Phi_{\mathfrak{z}}$ in the sense of Lyapunov, then we can find an interpolating $F \in H^\infty(E)$ of norm at most 1. The route we shall follow is similar, in certain respects, to the route followed in the proof of [4, Theorem 5.3] and is based, ultimately, on the commutant lifting approach to the classical Nevanlinna-Pick theorem pioneered by Sarason [9]. For this purpose, we need another way to express Lyapunov dominance that reflects the fact that the \mathfrak{z}_i 's involved all lie in $\mathcal{AC}(E, \sigma)$. The key tool in our approach is the notion of an induced representation for $\mathcal{T}_+(E)$ and the connection such representations have with the concept of absolute continuity. They are defined as follows: Let τ be a normal representation of M on the Hilbert space H_τ . Then we may induce τ to $\mathcal{F}(E)$, obtaining a normal representation $\tau^{\mathcal{F}(E)}$ of $\mathcal{L}(\mathcal{F}(E))$ on the Hilbert space $\mathcal{F}(E) \otimes_\tau H_\tau$. The restriction of $\tau^{\mathcal{F}(E)}$ to $\mathcal{T}_+(E)$, then, is called the *representation of $\mathcal{T}_+(E)$ induced by τ* . It is clearly an absolutely continuous representation of $\mathcal{T}_+(E)$, since $H^\infty(E)$ is contained in $\mathcal{L}(\mathcal{F}(E))$ by definition and $\tau^{\mathcal{F}(E)}$ is ultraweakly continuous. As we showed in [2], and will discuss in a moment, induced representations are the archetypical absolutely continuous representations. We continue to use the notation $\tau^{\mathcal{F}(E)}$ for its restrictions to $\mathcal{T}_+(E)$ and $H^\infty(E)$.

In [5] we develop at length the analogies between induced representations of $\mathcal{T}_+(E)$ and $H^\infty(E)$ and unilateral shifts. Indeed, a unilateral shift arises from an induced representation of $\mathcal{T}_+(E)$, where $M = E = \mathbb{C}$.

Definition 3.1. Let π be a faithful representation of M on H_π and assume that π has infinite multiplicity. Then $\pi^{\mathcal{F}(E)}$ is called *the universal induced representation* of $\mathcal{T}_+(E)$ and $H^\infty(E)$ determined by π .

Any two faithful π 's with infinite multiplicity give unitarily equivalent induced representations. Further, every induced representation of $\mathcal{T}_+(E)$ is unitarily equivalent to a subrepresentation of $\pi^{\mathcal{F}(E)}$ obtained by restricting $\pi^{\mathcal{F}(E)}$ to a subspace of the form $\mathcal{F}(E) \otimes_\pi \mathcal{K}$, where \mathcal{K} is a subspace of H_π that reduces π [2, Paragraphs 2.5 and 2.11]. This explains the terminology, allowing us to use the definite article. The representation π and the induced representation $\pi^{\mathcal{F}(E)}$ will be fixed for the remainder of this note.

We also shall extend the notation $\mathcal{I}(\sigma^E \circ \varphi, \sigma)$, and write $\mathcal{I}(\rho_1, \rho_2)$ for the set of operators $C : H_{\rho_1} \rightarrow H_{\rho_2}$ that intertwine ρ_1 and ρ_2 , where ρ_1 and ρ_2 are any two completely contractive representations of $\mathcal{T}_+(E)$. If ρ_i is written as $\mathfrak{z}_i \times \sigma_i$, $i = 1, 2$, then it is easy to see that an operator $C : H_{\rho_1} \rightarrow H_{\rho_2}$ lies in $\mathcal{I}(\rho_1, \rho_2)$ if and only if $C \in \mathcal{I}(\sigma_1, \sigma_2)$ and $\mathfrak{z}_2(I_E \otimes C) = C\mathfrak{z}_1$.

An important linkage among the universal induced representation, intertwiners, and absolute continuity is the following theorem.

Theorem 3.2. [2, Theorem 4.7] *A point $\mathfrak{z} \in \overline{\mathbb{D}(E, \sigma)}$ is absolutely continuous if and only if*

$$\bigvee \{ \text{Ran}(c) \mid c \in \mathcal{I}(\pi^{\mathcal{F}(E)}, \mathfrak{z} \times \sigma) \} = H_\sigma.$$

Further, for $F \in H^\infty(E)$, $\mathfrak{z} \in \mathcal{AC}(E, \sigma)$, and $c \in \mathcal{I}(\pi^{\mathcal{F}(E)}, \mathfrak{z} \times \sigma)$,

$$(3) \quad \widehat{F}(\mathfrak{z})c = c\pi^{\mathcal{F}(E)}(F).$$

Proof. The first assertion is explicitly in [2] as Theorem 4.7. The second assertion is easily checked on generators of $H^\infty(E)$ of the form $\varphi_\infty(a)$, $a \in M$, and T_ξ , $\xi \in E$. That is all that is necessary to check. \square

Recall that if $\mathfrak{z} \in \overline{\mathbb{D}(E, \sigma)}$, then \mathfrak{z}^* lies in the W^* -correspondence E^σ over $\sigma(M)'$. It therefore defines a completely positive map $\Theta_\mathfrak{z}$ on $\sigma(M)'$ by the formula

$$\Theta_\mathfrak{z}(a) = \langle \mathfrak{z}^*, a \cdot \mathfrak{z}^* \rangle = \mathfrak{z}(I_E \otimes a)\mathfrak{z}^*, \quad a \in \sigma(M)'.$$

Indeed, $\Theta_\mathfrak{z}$ is just a special case of the map $\Phi_\mathfrak{z}$ in the statement of Theorem 2.4. We are going to use the following theorem from [2] to obtain an alternate formulation of the complete Lyapunov dominance assertion in that theorem.

Theorem 3.3. [2, Theorem 4.6] *If $\mathfrak{z} \in \overline{\mathbb{D}(E, \sigma)}$, then an operator $q \in \sigma(M)'$ is a pure superharmonic operator for $\Theta_\mathfrak{z}$ if and only if q can be written as $q = cc^*$ for an element $c \in \mathcal{I}(\pi^{\mathcal{F}(E)}, \mathfrak{z} \times \sigma)$.*

Corollary 3.4. *We adopt the notation of Theorem 2.4. The map Φ_W completely dominates $\Phi_\mathfrak{z}$ in the sense of Lyapunov if and only if the following condition is satisfied: For every integer $m \geq 1$, for every choice of function $l : \{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, n\}$, and for any choice of m operators $c_j \in \mathcal{I}(\pi^{\mathcal{F}(E)}, \mathfrak{z}_{l(j)} \times \sigma)$ the operator matrix inequality*

$$(4) \quad (W_{l(i)}c_i c_j^* W_{l(j)}^*)_{i,j=1}^m \leq (c_i c_j^*)_{i,j=1}^m$$

is satisfied.

Proof. Fix m and a function $l : \{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, n\}$. Write $H_\sigma^{(m)}$ for the direct sum of m copies of H_σ and let σ_m be the inflated representation of M on $H_\sigma^{(m)}$, i.e., $\sigma_m(a) := \text{diag}\{a, a, \dots, a\}$. Then $\sigma_m(M)' = M_m(\sigma(M)')$. Also, write $E^{(m)}$ for the direct sum of copies of E , which is also a W^* -correspondence over M in the obvious way, and set $\tilde{\mathfrak{z}} := (\mathfrak{z}_{l(1)}, \mathfrak{z}_{l(2)}, \dots, \mathfrak{z}_{l(m)})$. Then we may view $\tilde{\mathfrak{z}}$ as a map from $E^{(m)} \otimes_{\sigma_m} H_\sigma^{(m)} = (E \otimes_\sigma H_\sigma)^{(m)}$ to $H_\sigma^{(m)}$, which clearly belongs to $\mathcal{I}(\sigma_m^{E^{(m)}} \circ \varphi, \sigma_m)$. Consequently, $\tilde{\mathfrak{z}}$ defines a completely positive map $\Theta_{\tilde{\mathfrak{z}}}$ on $\sigma_m(M)' = M_m(\sigma(M)')$, and it is easy to see that

$$\Theta_{\tilde{\mathfrak{z}}}((b_{l(i), l(j)}))_{i,j=1}^m = \left(\mathfrak{z}_{l(i)}(I_E \otimes b_{l(i), l(j)}) \mathfrak{z}_{l(j)}^* \right), \quad (b_{l(i), l(j)})_{i,j=1}^m \in M_m(\sigma(M)').$$

Moreover, by Theorem 3.3, the pure superharmonic elements $M_m(\sigma(M)')$ for $\Theta_{\tilde{\mathfrak{z}}}$ are of the form $(c_i c_j^*)_{i,j=1}^m$, where $c_i \in \mathcal{I}(\pi^{\mathcal{F}(E)}, (\mathfrak{z}_{l(i)} \times \sigma))$. On the other hand, the W_i 's may be used to define the completely positive map $\Psi_{W,l}$ on $M_m(\sigma(M)')$ by the formula

$$\Psi_{W,l}((b_{l(i), l(j)})) := \left(W_{l(i)} b_{l(i), l(j)} W_{l(j)}^* \right), \quad (b_{l(i), l(j)})_{i,j=1}^m \in M_m(\sigma(M)').$$

The inequality 4 is the statement that the condition of the corollary is equivalent to the assertion that $\Psi_{W,l}$ dominates $\Theta_{\tilde{\mathfrak{z}}}$ in the sense of Lyapunov for each choice of m and l . It is now evident that the condition of the corollary implies that Φ_W completely dominates $\Phi_{\tilde{\mathfrak{z}}}$ in the sense of Lyapunov by choosing m and l judiciously. On the other hand, if Φ_W completely dominates $\Phi_{\tilde{\mathfrak{z}}}$, then given m and l , one can clearly choose a k so that the domination of $(\Phi_{\tilde{\mathfrak{z}}})_k$ by $(\Phi_W)_k$ in the sense of Lyapunov gives the desired inequalities of the condition for that m and l . \square

In order to follow the commutant lifting approach pioneered by Sarason, we require the description of the commutant of $\pi^{\mathcal{F}(E)}(H^\infty(E))$ that we developed in [4]. The description there works for any induced representation, but we formulate it here specifically for $\pi^{\mathcal{F}(E)}$.

Theorem 3.5. [4, Theorem 3.9] *Write ι for the identity representation of $\pi(M)'$ on H_π , and let τ be the induced representation of $\mathcal{L}(E^\pi)$ acting on $\mathcal{F}(E^\pi) \otimes_\iota H_\pi$, i.e., let $\tau = \iota^{\mathcal{F}(E^\pi)}$. Then the map $U : \mathcal{F}(E^\pi) \otimes_\iota H_\pi \rightarrow \mathcal{F}(E) \otimes_\pi H_\pi$, defined by the formula*

$$(5) \quad U(\xi_1 \otimes \xi_2 \otimes \dots \otimes \xi_n \otimes h) := (I_{E^{\otimes(n-1)}} \otimes \xi_1)(I_{E^{\otimes(n-2)}} \otimes \xi_2) \dots (I_E \otimes \xi_{n-1}) \xi_n(h),$$

$\xi_1 \otimes \xi_2 \otimes \dots \otimes \xi_n \otimes h \in (E^\pi)^{\otimes n} \otimes_\iota H_\pi$, is a Hilbert space isomorphism and

$$U\tau(H^\infty(E^\pi))U^* = \pi^{\mathcal{F}(E)}(H^\infty(E))'.$$

Likewise, $U^*\pi^{\mathcal{F}(E)}(H^\infty(E))U = \tau(H^\infty(E^\pi))'$, and the double commutant relations hold:

$$\pi^{\mathcal{F}(E)}(H^\infty(E))'' = \pi^{\mathcal{F}(E)}(H^\infty(E)), \quad \text{and} \quad \tau(H^\infty(E^\pi))'' = \tau(H^\infty(E^\pi)).$$

We are now ready to show how the complete domination of $\Phi_{\tilde{\mathfrak{z}}}$ in the sense of Lyapunov by Φ_W implies that we can interpolate the W 's at the \mathfrak{z} 's in Theorem 2.4.

Lemma 3.6. *Let*

$$\mathcal{M} = \overline{\text{span}}\{U^*c^*h \mid h \in H_\sigma, c \in \mathcal{I}(\pi^{\mathcal{F}(E)}, \mathfrak{z}_i \times \sigma), 1 \leq i \leq n\}.$$

Then \mathcal{M} is a closed subspace of $\mathcal{F}(E^\pi) \otimes_\iota H_\pi$ that is invariant under $\tau(H^\infty(E^\pi))^$.*

Proof. For $X \in \tau(H^\infty(E^\pi))$, $U\tau(X)U^*$ lies in the commutant of $\pi^{\mathcal{F}(E)}(H^\infty(E))$ by Theorem (3.5). Consequently $cU\tau(X)U^* \in \mathcal{I}(\pi^{\mathcal{F}(E)}, \mathfrak{z}_i \times \sigma)$ for every $c \in \mathcal{I}(\pi^{\mathcal{F}(E)}, \mathfrak{z}_i \times \sigma)$. But then $\tau(X)^*U^*c^*h = U^*(U\tau(X)^*U^*)c^*h = U^*(cU\tau(X)U^*)^*h$ lies in \mathcal{M} for all $U^*c^*h \in \mathcal{M}$. \square

Lemma 3.7. *The correspondence, $U^*c^*h \rightarrow U^*c^*W_i^*$, $c \in \mathcal{I}(\pi^{\mathcal{F}(E)}, \mathfrak{z}_i \times \sigma)$, defined on the generators of \mathcal{M} extends to a well-defined contraction operator on \mathcal{M} , say R , if and only if for every integer $m \geq 1$, for every choice of function $l : \{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, n\}$, and for every choice of m operators $c_j \in \mathcal{I}(\pi^{\mathcal{F}(E)}, \mathfrak{z}_{l(j)} \times \sigma)$ the operator matrix inequality*

$$(W_{l(i)}c_ic_j^*W_{l(j)}^*)_{i,j=1}^m \leq (c_ic_j^*)_{i,j=1}^m$$

is satisfied. In this event, R commutes with the restriction of $\tau(H^\infty(E^\pi))^$ to \mathcal{M} .*

Proof. A linear combination of generators of \mathcal{M} is a vector of the form $k = \sum_{j=1}^m U^*c_j^*h_j$, where $c_j \in \mathcal{I}(\pi^{\mathcal{F}(E)}, \mathfrak{z}_{l(j)} \times \sigma)$ for some m and function $l : \{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, n\}$. Since

$$\|k\|^2 = \sum_{j,i} \langle c_ic_j^*h_j, h_i \rangle,$$

while

$$\left\| \sum_{j=1}^m U^*c_j^*W_{l(j)}^*h_j \right\|^2 = \sum_{i,j} \langle W_{l(i)}c_ic_j^*W_{l(j)}^*h_j, h_i \rangle,$$

the first assertion is immediate. But the second is also immediate since R is “right multiplication” by W_i^* on a generator of the form U^*c^*h , $c \in \mathcal{I}(\pi^{\mathcal{F}(E)}, \mathfrak{z}_i \times \sigma)$, i.e., $RU^*c^*h = U^*c^*W_i^*h$, while the restriction of $\tau(X)^*$ to \mathcal{M} acts by left multiplication for all $X \in H^\infty(E^\pi)$: $\tau(X)^*U^*c^*h = U^*(U\tau(X)^*U^*)c^*h$. \square

Since \mathcal{M} is invariant for $\tau(H^\infty(E^\pi))^*$, we obtain an ultra weakly continuous completely contractive representation ρ of $H^\infty(E^\pi)$ on \mathcal{M} by compressing $\tau(H^\infty(E^\pi))$ to \mathcal{M} , i.e.,

$$\rho(X) := P_{\mathcal{M}}\tau(X)|_{\mathcal{M}}, \quad X \in H^\infty(E^\pi).$$

Since τ is isometric in the sense of [6] and since R^* commutes with $\rho(H^\infty(E^\pi))$, we may apply our commutant lifting theorem [6, Theorem 4.4] to conclude that there is an operator $Y \in B(\mathcal{F}(E^\pi) \otimes_\iota H_\pi)$ of norm at most one such that $P_{\mathcal{M}}Y|_{\mathcal{M}} = R^*$, $Y\mathcal{M}^\perp \subseteq \mathcal{M}^\perp$, and Y commutes with $\tau(H^\infty(E^\pi))$ (see [2, Theorem 2.6], also). By Theorem 3.5, there is an $F \in H^\infty(E)$, $\|F\| \leq 1$, such that $Y = U^*\pi^{\mathcal{F}(E)}(F)U$. We conclude from the properties of Y and the definition of R that

$$U^*\pi^{\mathcal{F}(E)}(F)^*c^*h = (U^*\pi^{\mathcal{F}(E)}(F)^*U)U^*c^*h = Y^*U^*c^*h = RU^*c^*h = U^*c^*W_i^*h$$

for all $c \in \mathcal{I}(\pi^{\mathcal{F}(E)}, \mathfrak{z}_i \times \sigma)$. This, in turn, implies that

$$c\pi^{\mathcal{F}(E)}(F) = W_ic$$

for all such c . But $c\pi^{\mathcal{F}(E)}(F) = \widehat{F}(\mathfrak{z}_i)c$ for all $c \in \mathcal{I}(\pi^{\mathcal{F}(E)}, \mathfrak{z}_i \times \sigma)$, by equation (3) in Theorem 3.2. Therefore,

$$\widehat{F}(\mathfrak{z}_i)c = W_ic$$

for all i and all $c \in \mathcal{I}(\pi^{\mathcal{F}(E)}, \mathfrak{z}_i \times \sigma)$. However, by hypothesis, all the \mathfrak{z}_i lie in $\mathcal{AC}(E, \sigma)$. Consequently, by the first assertion of Theorem 3.2, the closed span of the ranges of the c 's in $\mathcal{I}(\pi^{\mathcal{F}(E)}, \mathfrak{z}_i \times \sigma)$ is all of H_σ , for every i . We conclude that $\widehat{F}(\mathfrak{z}_i) = W_i$ for every i . This completes the proof that if Φ_W completely dominates $\Phi_{\mathfrak{z}}$ in the sense of Lyapunov, then there is an $F \in H^\infty(E)$ that interpolates W_i at \mathfrak{z}_i .

Proof of the Converse. Part of the argument just given is reversible. Suppose F is an element of $H^\infty(E)$ of norm at most one such that $\widehat{F}(\mathfrak{z}_i) = W_i$, $i = 1, 2, \dots, n$. Then for each $c \in \mathcal{I}(\pi^{\mathcal{F}(E)}, \mathfrak{z}_i \times \sigma)$

$$c\pi^{\mathcal{F}(E)}(F) = \widehat{F}(\mathfrak{z}_i)c = W_ic,$$

by equation (3). But then

$$(U^*\pi^{\mathcal{F}(E)}(F)^*U)U^*c^* = U^*c^*W_i^*$$

for all $c \in \mathcal{I}(\pi^{\mathcal{F}(E)}, \mathfrak{z}_i \times \sigma)$. Since the norm of F is at most 1 we conclude from Lemma 3.7 that for every integer $m \geq 1$, for every choice of function $l : \{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, n\}$, and for every choice of m operators $c_j \in \mathcal{I}(\pi^{\mathcal{F}(E)}, \mathfrak{z}_{l(j)} \times \sigma)$ the operator matrix inequality

$$(W_{l(i)}c_ic_j^*W_{l(j)}^*)_{i,j=1}^m \leq (c_ic_j^*)_{i,j=1}^m$$

is satisfied. So by Corollary 3.4, we conclude that Φ_W completely dominates $\Phi_{\mathfrak{z}}$ in the sense of Lyapunov. \square

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